NMSU MATH PROBLEM OF THE WEEK

Solution to Problem 9

Fall 2024

Problem 9

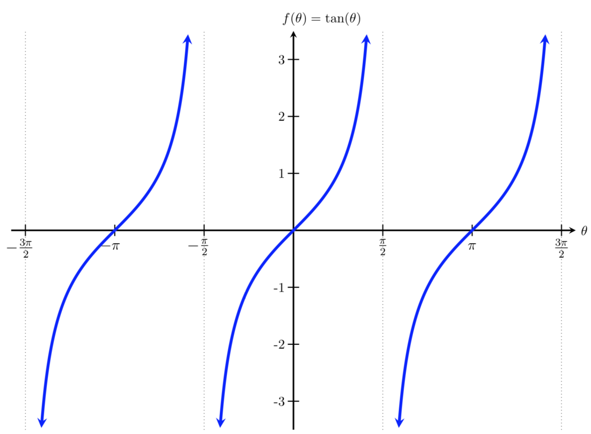
Let $\mathbf{S}$ be a set of any 4 distinct real numbers. Can one always find two distinct numbers $a, b \in \mathbf{S}$ such that

$$0 < \frac{a-b}{1+ab} \leq 1?$$

If your answer is yes, then give a proof. If your answer is no, provide a counterexample.

Solution. The statement is TRUE. Before we delve into the proof let us try to construct a counterexample.

Recall that that the trigonometric function $\tan(\theta)$ takes values from $-\infty$ to $+\infty$ as θ varies in $(\frac{\pi}{2}, \frac{\pi}{2})$:



Thus for any set with four distinct real numbers $\mathbf{S} = \{s_1, s_2, s_3, s_4\}$, we may write

$$s_1 = \tan(\theta_1), s_2 = \tan(\theta_2), s_3 = \tan(\theta_3), s_4 = \tan(\theta_4)$$

where $\theta_1, \theta_2, \theta_3$ and θ_4 are distinct numbers in $(-\frac{\pi}{2}, \frac{\pi}{2})$. Now note,

$$\tan(\theta_i - \theta_j) = \frac{\tan(\theta_i) - \tan(\theta_j)}{1 + \tan(\theta_i)\tan(\theta_j)} = \frac{s_i - s_j}{1 + s_i s_j}.$$

Since $0 < \tan(\theta) \leq 1$ whenever $\theta \in (0, \frac{\pi}{2})$, we must choose $\theta_1 < \theta_2 < \theta_3 < \theta_4$ such that $\theta_{i+1} - \theta_i > \frac{\pi}{4} = .25\pi$. This can be easily arranged. For instance, we may set

$$\theta_1 = -.45\pi, \theta_2 = -.15\pi, \theta_3 = .15\pi, \theta_4 = .45\pi.$$

However, if we analyze the set $\mathbf{S} = \{s_1, s_2, s_3, s_4\}$, where

$$s_1 = \tan(-.45\pi) = -6.313, s_2 = \tan(-.15\pi) = -.509,$$

$$s_3 = \tan(.15\pi) = .509, s_4 = \tan(.45\pi) = 6.313$$

then

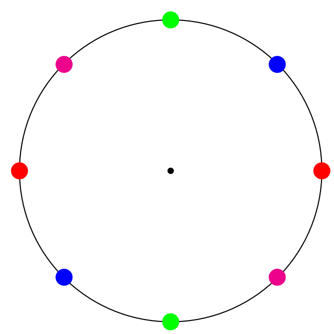
$$\frac{s_1 - s_4}{1 + s_1 s_4} = \frac{-12.616}{1 - (6.313)^2}$$

belongs to the interval $(0, 1]$. Thus, the set $\mathbf{S}$ is not quite a counterexample! This is a consequence of the fact that

$$\tan(\theta) = \tan(\theta + \pi).$$

Explictly, $\tan(-.45\pi) = \tan(-.45\pi + \pi) = \tan(.55\pi)$, and $(\theta_1 + \pi) - \theta_4 = .55\pi - .45\pi < \frac{\pi}{4}$.

Now we move towards the proof of the statement. From the above example, we learn that for each θ_i, we must consider the pair $\{\theta_i, \theta_i + \pi\}$ which represent a pair of antipode points on an unit circle.



To produce a counterexample, we require 4 pairs of antipode points on a circle, such that all any two points are at an angle greater than $\frac{\pi}{4} = 45°$. This is impossible by pigeonhole principle. Hence, the result.